ON THE STATIONARY DISTRIBUTION OF REFLECTED BROWNIAN MOTION IN A NON-CONVEX WEDGE

GUY FAYOLLE, SANDRO FRANCESCHI, AND KILIAN RASCHEL

Abstract. We study the stationary reflected Brownian motion in a non-convex wedge, which, compared to its convex analogue model, has been much rarely analyzed in the probabilistic literature. We prove that its stationary distribution can be found by solving a two dimensional vector boundary value problem (BVP) on a single curve for the associated Laplace transforms. The reduction to this kind of vector BVP seems to be new in the literature. As a matter of comparison, one single boundary condition is sufficient in the convex case. When the parameters of the model (drift, reflection angles and covariance matrix) are symmetric with respect to the bisector line of the cone, the model is reducible to a standard reflected Brownian motion in a convex cone. Finally, we construct a one-parameter family of distributions, which surprisingly provides, for any wedge (convex or not), one particular example of stationary distribution of a reflected Brownian motion.

1. Introduction

1.1. **Context and motivations.** Since the introduction of the reflected Brownian motion in the eighties [20, 19, 34, 36], the mathematical community has shown a constant interest in this topic. Typical questions deal with the recurrence of the process, the absorption at the corner of the wedge, the existence and computation of stationary distributions... We refer for more details to the introduction of [17].

Generally speaking, an obliquely reflected Brownian motion in a two-dimensional wedge of opening angle $\beta \in (0, 2\pi)$ is defined by its drift $\mu \in \mathbb{R}^2$ and two reflection angles $(\delta, \varepsilon) \in (0, \pi)^2$, see Figures 1.4, 2.1 and 5.1 for a few examples. The covariance matrix is taken to be the identity. A suitable linear transform allows to reduce the whole range of parameter angles $\beta \in (0, 2\pi)$ to only three cases: the quarter plane (when $\beta \in (0, \pi)$), the three-quarter plane (when $\beta \in (\pi, 2\pi)$) and the limiting half-plane case $\beta = \pi$. Doing so, the covariance matrix is nolonger the identity but instead has the general form (2.1).

While the early articles [34, 36] most dealt with the general case $\beta \in (0, 2\pi)$, the subcase of convex cones $\beta \in (0, \pi]$ has attracted much more attention [20, 19, 14,

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13, 1, 7, 5, 6, 16, 17, 3]; we have identified at least three reasons for that. First, one initial motivation was to approximate queueing systems in a dense traffic regime [18], which are typically obtained from random walks in the (convex) quarter plane. Second, the Laplace transform turns out to be a very useful tool in these problems; to make this function converge we need to have a convex cone. Finally, because there are already several parameters defining reflected Brownian motion (drift, reflection angles and opening angle), we feel that non-convex cones have sometimes been taken away, in order to reduce the number of cases to consider: for instance, regarding transience and recurrence criteria, only the convex case has been established in [23], while close arguments should also cover the non-convex case.

> *In this article, our main objective is the study of recurrent reflected Brownian motion in the non-convex case $\beta \in (\pi, 2\pi)$: we shall introduce complex analysis techniques to characterize the Laplace transform of the stationary distribution.*

Let us present five motivations to the present work. Our first goal is to complete the literature and to show how, in this more complicated non-convex setting, one can solve the problem of finding the stationary distribution. Our techniques could also be applied to the transient case, for example to analyse Green functions or absorption probabilities (see [15, 9] for the convex case); however, we do not tackle these problems here.

Our second motivation is provided by the discrete framework of random walks (or queueing networks). Indeed, in the same way as in the quarter plane, reflected Brownian motion has been introduced to study scaling limits of large queueing networks (see Figure 1.1), a Brownian model in a non-convex cone could approximate discrete random walks on a wedge having obtuse angle (see Figure 1.2 for a concrete example). Such random walks have an independent interest and have already been studied in a number of cases: see [2, 29, 8] in the combinatorial literature and [33, 27] for more probability inclined works.

Our third motivation is to develop an analytic method, which turns out to be particularly useful in a number of contexts. This method was invented by Fayolle, Iasnogorodski and Malyshev in the seventies, see [25, 10, 11]; at that time, the principal motivation was to study the stationary distribution of ergodic reflected random walks in a quadrant. The main idea is to state a functional equation satisfied by the associated generating functions and to reduce it to certain boundary value problems, which after analysis happen to be solvable in closed form. This approach has been applied to the framework of Brownian diffusions in a quadrant [14, 13, 1], to symmetric random walks in a three-quarter plane [29, 33], but never to the present setting of diffusions in non-convex wedges. From this technical point of view, the present work will bring the following novelty: we will prove that our problem is generically reducible to a system of two boundary value problems (as a matter of comparison, only one single boundary value problem is needed in the convex case [17]).

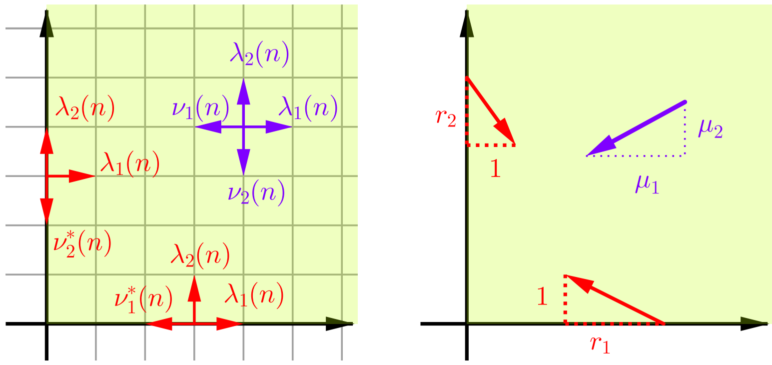

FIGURE 1.1. Scaling limit of some queueing systems towards reflected Brownian motion. Left picture: transition rates of a random walk (two coupled processors). Taking $\lambda_i(n), \nu_i(n) \to \frac{1}{2}$, $\sqrt{n}(\lambda_i - \nu_i) \to \mu_i$ and $\nu_i^*(n) \to \frac{r_i+1}{2}$, the discrete process converges to the reflected Brownian motion with parameters described as on the right picture (with identity covariance matrix). See [30] for the original proof.

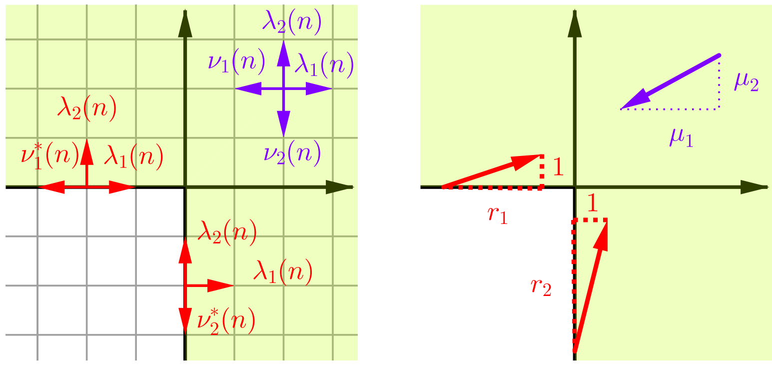

FIGURE 1.2. For the exact same reasons as for Figure 1.1, in the three-quarter plane, the discrete model on the left picture converges to the reflected Brownian motion on the right display.

Next, we aim at initiating the study of piecewise inhomogeneous Brownian models in cones of $\mathbb{R}^d$. To take a concrete example, consider a half-plane and view it as the union of two quarter planes glued along one half-axis (see Figure 1.3, leftmost picture). Then the process behaves as follows: in each quarter plane, its evolution is governed by a Brownian motion (with possible different drifts and covariance matrices); the process can pass from one quadrant to the other one through the porous interface; on the remaining boundaries, it is reflected in a standard way. Another example would consist in dividing

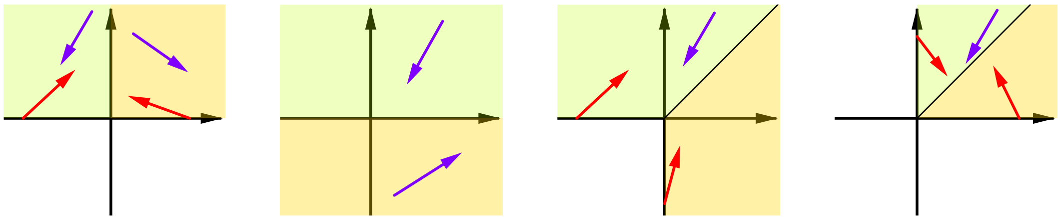

FIGURE 1.3. Different models of (non-)reflected inhomogeneous Brownian motions in various cones of $\mathbb{R}^2$. Blue arrows represent drift vectors and red arrows stand for the reflection vectors on the boundary axes. In the second picture, when the two drifts are opposite, the vertical component is called a bang-bang Brownian motion.

the plane into two half-planes, as on Figure 1.3, left. This model may be viewed as a two-dimensional generalization of the so-called bang-bang process on $\mathbb{R}$, as studied in [32].

Piecewise inhomogeneous Brownian motions are related to our obtuse angle model as follows: splitting the three-quarter plane into two convex wedges (see the right display on Figure 1.3, or Figure 3.1) and performing simple linear transformations, our model turns out to be equivalent to the inhomogeneous domain described above.

These inhomogeneous models are reminiscent from a well-known model in queuing theory, known as the JSQ (for "join the shortest queue") model, see [11, Chap. 10] or [24]. In this model, the quarter plane is divided into two octants ($\pi/8$-wedges) and the random walk obeys to different (very specific rules) according to the octant. See the rightmost picture on Figure 1.3. The techniques developed in this paper offer a potential approach to solve this (asymmetric) Brownian JSQ model.

Our fifth and final motivation is to provide tools leading to a comparative study of reflected Brownian motion in convex and non-convex cones. Does this model admit a kind of phase transition around the critical angle $\beta = \pi$? Some results in our paper tend to show that this is the case: while reflected Brownian motion in a convex cone may be studied with one single boundary value problem, two analogue problems are needed in the non-convex case. On the other hand, we also bring some evidence that the model has a smooth behavior at $\beta = \pi$: we are able to construct a one-parameter family of stationary distributions, whose formula is valid for any $\beta \in (0, 2\pi)$ and, surprisingly, is independent of β! While we will leave the question of phase transition as an open problem, let us conclude with the expression of the density (written in polar coordinates) of this remarkable family:

$$\pi(r,t) = \frac{C}{\sqrt{r}} \cos\left(\frac{t}{2}\right) e^{-2r|\mu|\cos^2\left(\frac{t}{2}\right)}, \tag{1.1}$$

where $|\mu|$ stands for the norm of the drift and C is a normalization constant; see Figure 1.4.

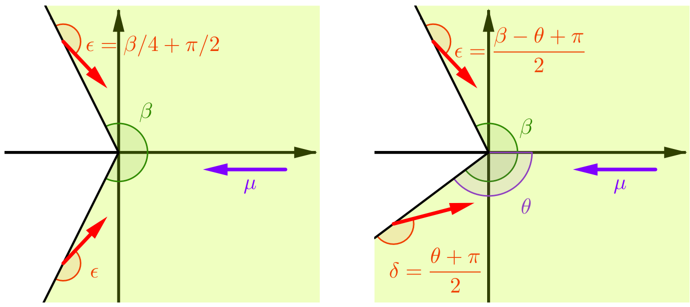

FIGURE 1.4. Parameters of the model leading to the remarkable stationary distribution (1.1). A priori, no symmetry assumption is done on the parameters (the model on the left is symmetric, contrary to the one on the right). Similarly, no convex hypothesis is done on the cone. The formula (1.1) has been obtained in [3] in the convex case and in [18, §9] in a more restrictive case, and we observe here that the same formula holds for any value of the opening angle β. Up to our knowledge, (1.1) is the unique example for which the stationary distribution density is known in closed form for a non-convex cone.

1.2. **Main results.** To conclude this introduction, we present the structure of the paper and our main results.

- Section 2: definition of the model, statement of the recurrence conditions and introduction of the stationary distribution
- Section 3: Proposition 2.1 on the classical basic adjoint relationship (characterizing the stationary distribution) and Proposition 3.1 on a system of two functional equations (the 3/4 plane is split into two convex cones of angle $3\pi/8$, and one equation is stated for each domain)
- Section 4: general study of the asymmetric case. Various statements on the kernel, meromorphic continuation of the unknown Laplace transforms, reduction to a Riemann-Hilbert vector boundary value problem (Theorem 4.4), relation with a Fredholm integral equation
- Section 5: general study of the symmetric case. Equivalence with a standard Brownian motion in a quarter plane, resolution and examples

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2. Semimartingale reflected Brownian motion avoiding a quarter plane

2.1. **Definition of the process.** We denote the three-quarter plane as

$$S \stackrel{\text{def}}{=} \{(z_1, z_2) \in \mathbb{R}^2 : z_1 \geqslant 0 \text{ or } z_2 \geqslant 0\}.$$

The parameters of the model are the drift $\mu = (\mu_1, \mu_2)$, the reflection vectors $R_1 = (r_1, 1)$ and $R_2 = (1, r_2)$, and the covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_1 & \rho \\ \rho & \sigma_2 \end{pmatrix}, \tag{2.1}$$

see Figure 2.1. Throughout this study, Σ will be assumed to be elliptic, i.e., $\sigma_1\sigma_2 - \rho^2 > 0$.

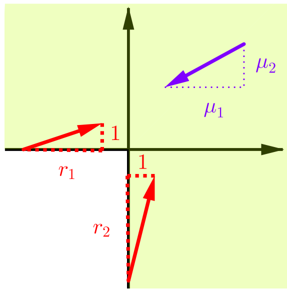

FIGURE 2.1. In green color, the three-quarter plane S, in blue the drift μ and in red the reflection vectors R_1 and R_2.

More specifically, we define the obliquely reflected Brownian motion $Z_t = (Z_t^1, Z_t^2)$ in the three-quarter plane S as follows:

$$\begin{cases} Z_t^1 \overset{\text{def}}{=} Z_0^1 + W_t^1 + \mu_1 t + r_1 L_t^1 + L_t^2, \\ Z_t^2 \overset{\text{def}}{=} Z_0^2 + W_t^2 + \mu_2 t + L_t^1 + r_2 L_t^2, \end{cases} \tag{2.2}$$

where W_t is a planar Brownian motion of covariance Σ, L_t^1 is (up to a constant) the local time on the negative part of the abscissa ($z_1 \leqslant 0$) and L_t^2 is the local time on the negative part of the ordinate axis ($z_2 \leqslant 0$). In case of a zero drift, such a semimartingale definition of reflected Brownian motion is proposed in the reference paper [36]; it readily extends to our drifted case.

Throughout this paper, we assume that the process is positive recurrent and has a unique stationary distribution (or invariant measure). As it turns out, this is equivalent to

$$\mu_1 < 0 \quad \text{and} \quad \mu_2 < 0, \tag{2.3}$$

and

$$\mu_1 - r_1\mu_2 > 0 \quad \text{and} \quad \mu_2 - r_2\mu_1 > 0. \tag{2.4}$$

(In particular, one has $r_1 > 0$ and $r_2 > 0$.) We couldn't find any reference proving this statement; however, the same techniques as in [23] by Hobson and Rogers (proving necessary and sufficient conditions in the quadrant similar as (2.3) and (2.4)) could be used here. Figure 2.1 represents a case where the parameters satisfy both conditions (2.3) and (2.4).

Under conditions (2.3) and (2.4), we will denote by Π this probability measure and by π its density. We also define the boundary invariant measures by

$$\nu_1(A) = \mathbb{E}_\Pi \int_0^1 1_{A\times\{0\}}(Z_s)\mathrm{d}L_s^1 \quad \text{and} \quad \nu_2(A) = \mathbb{E}_\Pi \int_0^1 1_{\{0\}\times A}(Z_s)\mathrm{d}L_s^2.$$

The measure ν_1 has its support on $\{z_1 \leqslant 0\}$ and ν_2 has its support on $\{z_2 \leqslant 0\}$. We will also denote by $\nu_1(z_1)$ and $\nu_2(z_2)$ their respective densities.

Remark that a reflected Brownian motion in the three-quarter plane could be defined as well in the non-semimartingale case; motivations to consider these cases are proposed in [28].

2.2. **Basic adjoint relationship.** Our approach is based on the following identity, called basic adjoint relationship, which in the orthant case is proved in [4, 21].

Proposition 2.1. *If f is the difference of two convex functions in S and if all the integrals below converge, then*

$$\int_S \mathcal{G}f(z_1,z_2)\pi(z_1,z_2)\mathrm{d}z_1\mathrm{d}z_2 + \int_{-\infty}^0 R_1\cdot\nabla f(z_1,0)\nu_1(z_1)\mathrm{d}z_1 + \int_{-\infty}^0 R_2\cdot\nabla f(0,z_2)\nu_2(z_2)\mathrm{d}z_2 = 0,$$

where the generator is equal to

$$\mathcal{G}f = \frac{1}{2}\left(\sigma_1\frac{\partial^2 f}{\partial z_1^2} + 2\rho\frac{\partial^2 f}{\partial z_1\partial z_2} + \sigma_2\frac{\partial^2 f}{\partial z_2^2}\right) + \mu_1\frac{\partial f}{\partial z_1} + \mu_2\frac{\partial f}{\partial z_2}.$$

Proof. We apply the ItÃt'-Tanaka formula to the semimartingale Z_t, see Theorem 1.5 in [31, Chap. VI §1]. We obtain

$$f(Z_t) = f(Z_0) + \int_0^t \mathcal{G}f(Z_s)\mathrm{d}s + \int_0^t \nabla f(Z_s)\mathrm{d}W_s + \sum_{i\in\{1,2\}}\int_0^t R_i\cdot\nabla f(Z_s)\mathrm{d}L_s^i.$$

To conclude, we take the expectation over Π of the above equality. ■

Since we take f to be the difference of two convex functions, the first derivatives of f are defined as the left derivatives, and the second derivatives of f are understood in the sense of distributions.

3. The main functional equations

Our goal is to use the basic adjoint relationship of Proposition 2.1 to obtain a kernel equation for the Laplace transform of the stationary distribution. In the case of a convex cone, it is enough to take $f(z_1,z_2) = e^{xz_1+yz_2}$ to obtain the functional equation, see [5, 16, 17]. However, if the cone is not convex, these integrals will not converge. So we need to divide the three-quarter plane into two regions. We define the two following $\frac{3}{8}$-planes:

$$S_1 \stackrel{\text{def}}{=} \{(z_1, z_2) \in \mathbb{R}^2 : z_1 \leqslant z_2 \text{ and } z_2 \geqslant 0\}$$

and $S_2 \stackrel{\text{def}}{=} S \setminus S_1$, see Figure 3.1.

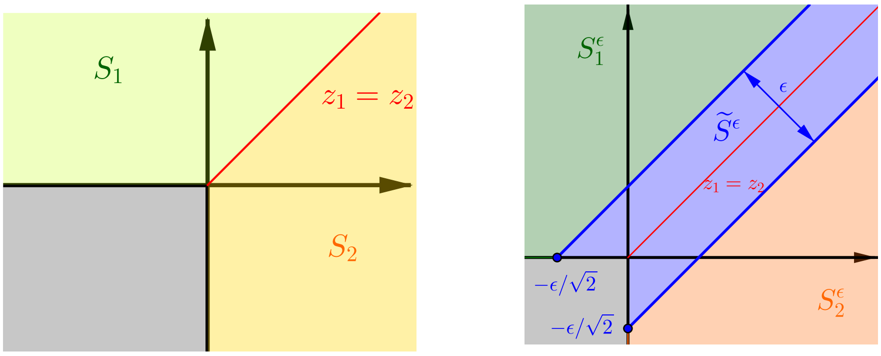

FIGURE 3.1. Left: the three-quarter plane divided in two sets, S_1 in green and S_2 in orange. Right: the three sets S_1^ε (in green color), S_2^ε (orange) and $\widetilde{S}^\varepsilon$ (blue).

Let us define the Laplace transform of the invariant measure π in S_1 by

$$L_1(x, y) \stackrel{\text{def}}{=} \int_{S_1} e^{xz_1 + yz_2} \pi(z_1, z_2) \mathrm{d}z_1 \mathrm{d}z_2, \tag{3.1}$$

the Laplace transform of π on the diagonal

$$m(x + y) \stackrel{\text{def}}{=} \int_0^\infty e^{(x+y)z} \pi(z, z) \mathrm{d}z,$$

the Laplace transform of the normal derivative of π on the diagonal

$$n(x + y) \stackrel{\text{def}}{=} \int_0^\infty e^{(x+y)z} \left(\frac{\partial \pi}{\partial z_1}(z, z) - \frac{\partial \pi}{\partial z_2}(z, z) \right) \mathrm{d}z, \tag{3.2}$$

and the Laplace transform of the boundary measure ν_1 on the abscissa

$$\ell_1(x) \stackrel{\text{def}}{=} \int_{-\infty}^0 e^{xz_1} \nu_1(z_1) \mathrm{d}z_1.$$

Introduce finally the constant

$$\theta \stackrel{\text{def}}{=} \frac{\sigma_1 + \sigma_2 - 2\rho}{2},$$

which is positive due to the ellipticity condition $\rho^2 - \sigma_1\sigma_2 < 0$.

The remainder of Section 3 is devoted to proving the following result:

Proposition 3.1 (Functional equation in S_1). *For all (x,y) in $\{\Re(x)\geqslant 0,\ \Re(x+y)\leqslant 0\}$, we have*

$$-K(x,y)L_1(x,y)= \\ k(x,y)m(x+y)+\theta n(x+y)+k_1(x,y)\ell_1(x)+(1-r_1)\nu_1(0)+(r_2-1)\nu_2(0),$$

where the kernel is defined by

$$K(x,y)\overset{\text{def}}{=}\frac{1}{2}\left(\sigma_1x^2+2\rho xy+\sigma_2y^2\right)+\mu_1x+\mu_2y, \tag{3.3}$$

while k and k_1 are polynomials of degree one in two variables given by

$$k(x,y)\overset{\text{def}}{=}\frac{\theta(y-x)}{2}+\frac{1}{2}(\sigma_2-\sigma_1)(x+y)+\mu_2-\mu_1,$$
$$k_1(x,y)\overset{\text{def}}{=}r_1x+y.$$

A symmetric functional equation holds on the domain S_2, it involves the functions m and n above, as well as

$$L_2(x,y)\overset{\text{def}}{=}\int_{S_2}e^{xz_1+yz_2}\pi(z_1,z_2)\mathrm{d}z_1\mathrm{d}z_2$$

and

$$\ell_2(y)\overset{\text{def}}{=}\int_{-\infty}^{0}e^{yz_2}\nu_2(z_2)\mathrm{d}z_2.$$

See (4.2) for the exact statement.

Proof of Proposition 3.1. Let us introduce the three following sets

$$S_1^\varepsilon\overset{\text{def}}{=}\{(z_1,z_2):z_2>z_1+\varepsilon/\sqrt{2}\text{ and }z_2\geqslant 0\},$$
$$S_2^\varepsilon\overset{\text{def}}{=}\{(z_1,z_2):z_1>z_2+\varepsilon/\sqrt{2}\text{ and }z_1\geqslant 0\}$$

and $\widetilde{S}^\varepsilon\overset{\text{def}}{=}S\setminus(S_1^\varepsilon\cup S_2^\varepsilon)$. Then, we define the function I_ε such that

$$I_\varepsilon(z_1,z_2)\overset{\text{def}}{=}\begin{cases}1 & \text{if } z\in S_1^\varepsilon,\\ \frac{z_2-z_1}{\sqrt{2}\varepsilon}+\frac{1}{2} & \text{if } z\in\widetilde{S}^\varepsilon,\\ 0 & \text{if } z\in S_2^\varepsilon.\end{cases} \tag{3.4}$$

From now on, we will often omit to note the variables (z_1,z_2). We have

$$\nabla I_\varepsilon=\left(\frac{\partial I_\varepsilon}{\partial z_1},\frac{\partial I_\varepsilon}{\partial z_2}\right)=\begin{cases}(0,0) & \text{if } z\in S_1^\varepsilon\cup S_2^\varepsilon,\\ \left(\frac{-1}{\sqrt{2}\varepsilon},\frac{1}{\sqrt{2}\varepsilon}\right) & \text{if } z\in\widetilde{S}^\varepsilon,\end{cases}$$

and, for all $z\in S$,

$$\frac{\partial^2I_\varepsilon}{\partial z_1^2}=\frac{\partial^2I_\varepsilon}{\partial z_2^2}=-\frac{\partial^2I_\varepsilon}{\partial z_1\partial z_2}=\frac{1}{\sqrt{2}\varepsilon}\big(\delta_{\varepsilon/\sqrt{2}}(z_1-z_2)-\delta_{-\varepsilon/\sqrt{2}}(z_1-z_2)\big), \tag{3.5}$$

where δ_a is the Dirac distribution in a. For the sake of brevity, we write

$$I'_\varepsilon \stackrel{\text{def}}{=} \frac{\partial I_\varepsilon}{\partial z_2} = -\frac{\partial I_\varepsilon}{\partial z_1} \qquad \text{and} \qquad I''_\varepsilon \stackrel{\text{def}}{=} \frac{\partial^2 I_\varepsilon}{\partial z_1^2} = \frac{\partial^2 I_\varepsilon}{\partial z_2^2} = -\frac{\partial^2 I_\varepsilon}{\partial z_1 \partial z_2}.$$

Let us take $f_\varepsilon \stackrel{\text{def}}{=} e^{xz_1+yz_2} I_\varepsilon$. Its first and second derivatives are equal to

$$\begin{aligned}
\frac{\partial f_\varepsilon}{\partial z_1} &= \left(xI_\varepsilon + \frac{\partial I_\varepsilon}{\partial z_1}\right) e^{xz_1+yz_2},\\
\frac{\partial f_\varepsilon}{\partial z_2} &= \left(yI_\varepsilon + \frac{\partial I_\varepsilon}{\partial z_2}\right) e^{xz_1+yz_2},\\
\frac{\partial^2 f_\varepsilon}{\partial z_1^2} &= \left(x^2 I_\varepsilon + 2x\frac{\partial I_\varepsilon}{\partial z_1} + \frac{\partial^2 I_\varepsilon}{\partial z_1^2}\right) e^{xz_1+yz_2},\\
\frac{\partial^2 f_\varepsilon}{\partial z_2^2} &= \left(y^2 I_\varepsilon + 2y\frac{\partial I_\varepsilon}{\partial z_2} + \frac{\partial^2 I_\varepsilon}{\partial z_2^2}\right) e^{xz_1+yz_2},\\
\frac{\partial f_\varepsilon}{\partial z_1 \partial z_2} &= \left(xyI_\varepsilon + x\frac{\partial I_\varepsilon}{\partial z_2} + y\frac{\partial I_\varepsilon}{\partial z_1} + \frac{\partial^2 I_\varepsilon}{\partial z_1 \partial z_2}\right) e^{xz_1+yz_2}.
\end{aligned}$$

Therefore, the generator at f_ε is given by

$$\mathcal{G} f_\varepsilon = \left(K(x,y)I_\varepsilon + \Big(\frac{\partial K}{\partial y} - \frac{\partial K}{\partial x}\Big) I'_\varepsilon + \frac{1}{2}\Big(\frac{\partial^2 K}{\partial x^2} + \frac{\partial^2 K}{\partial y^2} - 2\frac{\partial^2 K}{\partial x \partial y}\Big) I''_\varepsilon\right) e^{xz_1+yz_2},$$

that is,

$$\mathcal{G} f_\varepsilon = \\ \left(K(x,y)I_\varepsilon + \big(\sigma_2 y - \sigma_1 x - \rho(y-x) + \mu_2 - \mu_1\big) I'_\varepsilon + \frac{1}{2}\left(\sigma_1 + \sigma_2 - 2\rho\right) I''_\varepsilon\right) e^{xz_1+yz_2}.$$

We also have

$$\begin{aligned}
R_1 \cdot \nabla f_\varepsilon(z_1, 0) &= \big((r_1 x + y) I_\varepsilon + (1 - r_1) I'_\varepsilon\big) e^{xz_1},\\
R_2 \cdot \nabla f_\varepsilon(0, z_2) &= \big((x + r_2 y) I_\varepsilon + (r_2 - 1) I'_\varepsilon\big) e^{yz_2}.
\end{aligned}$$

Now we apply the basic adjoint relationship of Proposition 2.1 to f_ε (which can be written as the difference of two convex functions); since all integrals converge, we obtain

$$\begin{aligned}
0 = K(x,y) &\int_S I_\varepsilon(z_1,z_2) e^{xz_1+yz_2} \pi(z_1,z_2) \mathrm{d}z_1 \mathrm{d}z_2\\
&+ \big(\sigma_2 y - \sigma_1 x - \rho(y-x) + \mu_2 - \mu_1\big) \int_S I'_\varepsilon(z_1,z_2) e^{xz_1+yz_2} \pi(z_1,z_2) \mathrm{d}z_1 \mathrm{d}z_2\\
&+ \frac{1}{2}\left(\sigma_1 + \sigma_2 - 2\rho\right) \int_S I''_\varepsilon(z_1,z_2) e^{xz_1+yz_2} \pi(z_1,z_2) \mathrm{d}z_1 \mathrm{d}z_2\\
&+ (r_1 x + y) \int_{-\infty}^0 I_\varepsilon(z_1, 0) e^{xz_1} \nu_1(z_1) \mathrm{d}z_1 + (1 - r_1) \int_{-\infty}^0 I'_\varepsilon(z_1, 0) e^{xz_1} \nu_1(z_1) \mathrm{d}z_1\\
&+ (x + r_2 y) \int_{-\infty}^0 I_\varepsilon(0, z_2) e^{yz_1} \nu_2(z_2) \mathrm{d}z_2 + (r_2 - 1) \int_{-\infty}^0 I'_\varepsilon(0, z_2) e^{yz_2} \nu_1(z_2) \mathrm{d}z_2.
\end{aligned} \tag{3.6}$$

Since $\lim\limits_{\varepsilon\to 0} I_\varepsilon = 1_{S_1}$, the dominated convergence theorem implies that:

$$\lim_{\varepsilon\to 0}\int_S I_\varepsilon(z_1,z_2)e^{xz_1+yz_2}\pi(z_1,z_2)\mathrm{d}z_1\mathrm{d}z_2 = \int_{S_1} e^{xz_1+yz_2}\pi(z_1,z_2)\mathrm{d}z_1\mathrm{d}z_2 = L(x,y),$$

$$\lim_{\varepsilon\to 0}\int_{-\infty}^0 I_\varepsilon(z_1,0)e^{xz_1}\nu_1(z_1)\mathrm{d}z_1 = \int_{-\infty}^0 e^{xz_1}\nu_1(z_1)\mathrm{d}z_1 = \ell_1(x),$$

$$\lim_{\varepsilon\to 0}\int_{-\infty}^0 I_\varepsilon(0,z_2)e^{yz_2}\nu_2(z_2)\mathrm{d}z_2 = 0.$$

We also have $\lim\limits_{\varepsilon\to 0} I'_\varepsilon(z_1,z_2) = \delta_0(z_2-z_1)$, then by continuity of π, ν_1 and ν_2, we obtain the limits:

$$\lim_{\varepsilon\to 0}\int_S I'_\varepsilon(z_1,z_2)e^{xz_1+yz_2}\pi(z_1,z_2)\mathrm{d}z_1\mathrm{d}z_2 = \int_0^\infty e^{(x+y)z}\pi(z,z)\mathrm{d}z = m(x+y),$$

$$\lim_{\varepsilon\to 0}\int_{-\infty}^0 I'_\varepsilon(z_1,0)e^{xz_1}\nu_1(z_1)\mathrm{d}z_1 = \nu_1(0),$$

$$\lim_{\varepsilon\to 0}\int_{-\infty}^0 I'_\varepsilon(0,z_2)e^{yz_2}\nu_2(z_2)\mathrm{d}z_2 = \nu_2(0).$$

Our next goal is to show that

$$\lim_{\varepsilon\to 0}\int_S I''_\varepsilon(z_1,z_2)e^{xz_1+yz_2}\pi(z_1,z_2)\mathrm{d}z_1\mathrm{d}z_2 = \frac{1}{2}n(x+y) + \frac{1}{2}(x-y)m(x+y).$$

To this end, we introduce the linear change of variable

$$(z_1,z_2) \overset{\text{def}}{=} \varphi(u,v) = \left(\frac{u-v}{\sqrt{2}},\frac{u+v}{\sqrt{2}}\right), \quad (u,v) = \varphi^{-1}(z_1,z_2) = \left(\frac{z_1+z_2}{\sqrt{2}},\frac{z_2-z_1}{\sqrt{2}}\right),$$

where $\det\varphi = 1$. Recall that for arbitrary constants a and c, we have $\delta_a(c\times\cdot) = \frac{1}{|c|}\delta_{a/c}(\cdot)$. So we deduce from (3.5) the equality $I''_\varepsilon\big(\varphi(u,v)\big) = \frac{1}{2}\big(\delta_{-\frac{\varepsilon}{2}} - \delta_{\frac{\varepsilon}{2}}\big)(v)$. Let us define

$$g(u,v) \overset{\text{def}}{=} e^{x\frac{u-v}{\sqrt{2}}+y\frac{u+v}{\sqrt{2}}}\pi\left(\frac{u-v}{\sqrt{2}},\frac{u+v}{\sqrt{2}}\right).$$

We have

$$\begin{aligned}\int_S I''_\varepsilon(z_1,z_2)e^{xz_1+yz_2}\pi(z_1,z_2)\mathrm{d}z_1\mathrm{d}z_2 &= \frac{1}{2\varepsilon}\int_{\varphi^{-1}(S)}\left(\delta_{-\frac{\varepsilon}{2}} - \delta_{\frac{\varepsilon}{2}}\right)(v)\ g(u,v)\mathrm{d}u\mathrm{d}v\\ &= \frac{1}{2\varepsilon}\int_{-\varepsilon/2}^\infty \left(g\Big(u,-\frac{\varepsilon}{2}\Big) - g\Big(u,\frac{\varepsilon}{2}\Big)\right)\mathrm{d}u\\ &\underset{\varepsilon\to 0}{\longrightarrow} \frac{-1}{2}\int_0^\infty \frac{\partial g}{\partial v}(u,0)\mathrm{d}u\\ &= \frac{1}{2}\int_0^\infty e^{(x+y)z}\left(\frac{\partial\pi}{\partial z_1} - \frac{\partial\pi}{\partial z_2}\right)(z,z)\mathrm{d}z\\ &\quad + \frac{1}{2}(x-y)\int_0^\infty e^{(x+y)z}\pi(z,z)\mathrm{d}z.\end{aligned}$$

Finally, letting $\varepsilon\to 0$ in (3.6) concludes the proof. ■

Notice that in the proof of Proposition 3.1, the particular expression (3.4) is not at all crucial: any similar function with the desired properties would have been suitable.

4. The general asymmetric case

4.1. **Sketch of the approach.** For the sake of brevity, we shall put

$$E \stackrel{\text{def}}{=} (1-r_1)\nu_1(0) - (1-r_2)\nu_2(0).$$

Then the two functional equations obtained in Section 3 (see in particular Proposition 3.1), corresponding to the regions S_1 and S_2 in the (z_1, z_2)-plane, see Figure 3.1, are simply rewritten as follows:

$$K(x,y)L_1(x,y) + k(x,y)m(x+y) + \theta n(x+y) + k_1(x,y)\ell_1(x) + E = 0, \tag{4.1}$$

in the region $\{\Re(x) \geqslant 0,\, \Re(x+y) \leqslant 0\}$;

$$K(x,y)L_2(x,y) - k(x,y)m(x+y) - \theta n(x+y) + k_2(x,y)\ell_2(y) - E = 0, \tag{4.2}$$

in the region $\{\Re(y) \geqslant 0,\, \Re(x+y) \leqslant 0\}$.

The main idea is to build a system where the new variables are defined in *one and the same region*, by means of a simple change of variables. Clearly, this operation has a cost, since there will be two different kernels, the positive side being they can be simultaneously analyzed starting from a common domain. The key milestones of the study are listed hereunder.

- *Make the meromorphic continuation of all functions in their respective (cut) complex planes.*
- *Construct a vectorial Riemann boundary value problem for the pair* (ℓ_1, ℓ_2).
- *For the sake of completeness, derive a Fredholm integral equation for* m.

4.2. **Functional equations and kernels.** Setting respectively

$$p = -x, \quad q = x + y, \quad \text{in Equation (4.1)},$$

and

$$p = -y, \quad q = x + y, \quad \text{in Equation (4.2)},$$

leads to the system

$$U(p,q)L_1(p,q) + A(p,q)m(q) + \theta n(q) + C(p,q)\ell_1(p) + E = 0, \tag{4.3}$$

$$V(p,q)L_2(p,q) + B(p,q)m(q) - \theta n(q) + D(p,q)\ell_2(p) - E = 0, \tag{4.4}$$

where both equations are a priori defined in the domain $\{\Re p \leqslant 0,\, \Re(q) \leqslant 0\}$, and

$$(4.5)\qquad \begin{cases} U(p,q) \overset{\text{def}}{=} \theta p^2 + \dfrac{\sigma_2}{2}q^2 + (\sigma_2-\rho)pq + (\mu_2-\mu_1)p + \mu_2 q, \\ V(p,q) \overset{\text{def}}{=} \theta p^2 + \dfrac{\sigma_1}{2}q^2 + (\sigma_1-\rho)pq + (\mu_1-\mu_2)p + \mu_1 q, \\ A(p,q) \overset{\text{def}}{=} \dfrac{\theta(2p+q)}{2} + \dfrac{(\sigma_2-\sigma_1)q}{2} + \mu_2 - \mu_1, \\ B(p,q) \overset{\text{def}}{=} \dfrac{\theta(2p+q)}{2} + \dfrac{(\sigma_1-\sigma_2)q}{2} + \mu_1 - \mu_2, \\ C(p,q) \overset{\text{def}}{=} (1-r_1)p + q, \\ D(p,q) \overset{\text{def}}{=} (1-r_2)p + q. \end{cases}$$

Notation. *For convenience and to distinguish between the two kernels, we shall add in a superscript position the letter u (resp. v) to any quantity related to the kernel $U(x,y)$ (resp. $V(x,y)$). Moreover, if a property holds both for u and v, the superscript letter is omitted ad libitum.*

Accordingly, the branches of the algebraic curve $U = 0$ (resp. $V = 0$) over the q-plane will be denoted by $P_i^u(q)$ (resp. $P_i^v(q)$), $i = 1, 2$. By definition, they are solutions to

$$(4.6)\qquad U(P_i^u(q), q) = 0 \quad\text{and}\quad V(P_i^v(q), q) = 0.$$

In particular, they are simple algebraic functions of order 2. Similarly, $Q_i^u(p)$ (resp. $Q_i^v(p)$) will stand for the branches over the p-plane, $i = 1, 2$.

Although we are mostly working under the stationary hypotheses (2.3) and (2.4), notice that Lemmas 4.1 and 4.2 below hold true for any value of the drift vector (μ_1, μ_2).

Lemma 4.1. *The functions $P_i^u(q)$ and $P_i^v(q)$, $i = 1, 2$, are analytic in the whole complex plane cut along $(-\infty, q_1] \cup [q_2, \infty)$, where the branch points $q_1 < 0$ and $q_2 > 0$ are the two real roots of the equation*

$$(4.7)\qquad (\rho^2 - \sigma_1\sigma_2)q^2 + 2[\mu_1(\rho-\sigma_2) + \mu_2(\rho-\sigma_1)]q + (\mu_1-\mu_2)^2 = 0.$$

Remarkably, q_1 and q_2 are the same for the two kernels U and V. Moreover:

- *The branches P_1^u and P_2^u are separated and satisfy*

$$(4.8)\qquad \begin{cases} \Re(P_1^u(ix)) \leqslant 0 \leqslant \Re(P_2^u(ix)), \quad \forall x \in \mathbb{R}, \\ \Re(P_1^u(q)) \leqslant \Re(P_2^u(q)), \quad \forall q \in \mathbb{C}, \\ P_1^u(0) = \min\left\{0, \dfrac{\mu_1-\mu_2}{2\theta}\right\}, \quad P_2^u(0) = \max\left\{0, \dfrac{\mu_1-\mu_2}{2\theta}\right\}. \end{cases}$$

  They map the cut $(-\infty, q_1]$ (resp. $[q_2, \infty)$) onto the right branch $\mathcal{H}_+^u$ (resp. left branch $\mathcal{H}_-^u$) of the hyperbola $\mathcal{H}^u$ with equation

$$(4.9)\qquad (\rho^2-\sigma_1\sigma_2)x^2 + (\sigma_2-\rho)^2y^2 + 2(\sigma_2\mu_1-\rho\mu_2)x + \frac{(\mu_2-\mu_1)(\sigma_2(\mu_1+\mu_2)-2\rho\mu_2)}{2\theta} = 0.$$

- *The branches P_1^v and P_2^v are separated and satisfy*

$$\begin{cases} \Re(P_1^v(ix)) \leqslant 0 \leqslant \Re(P_2^v(ix)), \quad \forall x \in \mathbb{R}, \\ \Re(P_1^v(q)) \leqslant \Re(P_2^v(q)), \quad \forall p \in \mathbb{C}, \\ P_1^v(0) = \min\left\{0, \dfrac{\mu_2-\mu_1}{2\theta}\right\}, \quad P_2^v(0) = \max\left\{0, \dfrac{\mu_2-\mu_1}{2\theta}\right\}. \end{cases} \tag{4.10}$$

They map the cut $(-\infty, q_1]$ (resp. $[q_2, \infty)$) onto the right branch $\mathcal{H}_+^v$ (resp. left branch $\mathcal{H}_-^v$) of the hyperbola $\mathcal{H}^v$ with equation

$$(\rho^2-\sigma_1\sigma_2)x^2+(\sigma_1-\rho)^2y^2+2(\sigma_1\mu_2-\rho\mu_1)x+\frac{(\mu_1-\mu_2)(\sigma_1(\mu_1+\mu_2)-2\rho\mu_1)}{2\theta} = 0. \tag{4.11}$$

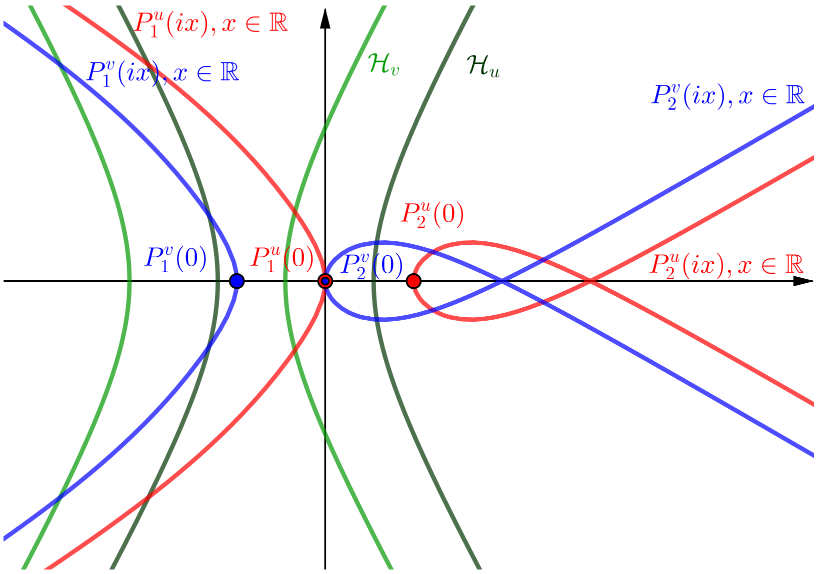

FIGURE 4.1. Illustration of Lemma 4.1 in the case where $\mu_1 > \mu_2$: curves $\{P_1^u(ix) : x \in \mathbb{R}\}$ and $\{P_2^u(ix) : x \in \mathbb{R}\}$ in red; curves $\{P_1^v(ix) : x \in \mathbb{R}\}$ and $\{P_2^v(ix) : x \in \mathbb{R}\}$ in blue; hyperbolas $\mathcal{H}_u$ and $\mathcal{H}_v$ in dark green and light green respectively.
Playing with the parameters is possible thanks to the following GeoGebra animation https://www.geogebra.org/m/phvjk35w

Proof. The branch points of $P(q)$ are the zeros of the discriminant of $U(p,q) = 0$ viewed as a polynomial in p; Equation (4.7) follows directly.

In order to prove (4.8), let $P(q)$ denote the multivalued algebraic function satisfying (4.6). Letting $q = ix$ with $x \in \mathbb{R}$ and $P(q) \overset{\text{def}}{=} \alpha + i\beta$ with real α, β, then separating real and imaginary parts, we obtain

$$\begin{cases} \theta\alpha^2 + (\mu_2 - \mu_1)\alpha - \big(\theta\beta^2 + (\sigma_2 - \rho)x\beta + \frac{\sigma_2}{2}x^2\big) = 0, \\ \beta(2\theta\alpha + \mu_2 - \mu_1) + x\big(\alpha(\sigma_2 - \rho) + \mu_2\big) = 0. \end{cases} \tag{4.12}$$

Then one checks that the first equation of (4.12), viewed as a polynomial in α, has two real roots with opposite sign. Indeed, the quadratic polynomial in β

$$\theta\beta^2 + (\sigma_2 - \rho)x\beta + \frac{\sigma_2}{2}x^2$$

is always positive, due to the ellipticity condition.

The second property of (4.8) is a direct application of the maximum modulus principle applied to the function $\exp P(q)$. More precisely, we look at the function $\exp P(q)$ on the domain $\mathbb{C} \setminus ((-\infty, q_1] \cup [q_2, \infty))$. Using the first property of (4.8), we deduce that for some values of q, one has

$$\big|\exp P_1(q)\big| \leqslant \big|\exp P_2(q)\big|. \tag{4.13}$$

On the other hand, on the cut $q \in (-\infty, q_1] \cup [q_2, \infty)$, the branches $P_1(q)$ and $P_2(q)$ are complex conjugate and thus $\big|\exp P_1(q)\big| = \big|\exp P_2(q)\big|$. Since the cut is the boundary of the cut plane, the maximum modulus principle entails that the inequality (4.13) holds true globally on $\mathbb{C}$.

The analytic expression (4.9) of the hyperbola follows from direct computations, see Lemma 5.10 and its proof for similar computations.

We note the pleasant symmetry of (4.7) with respect to the parameters. This is mainly due to the change of parameters from (x, y) to (p, q). As it will emerge later, that symmetry plays an important role in our analysis. The proof of the lemma is complete. ∎

Quite analogous properties hold for $Q_i^u(p)$ and $Q_i^v(p)$, but now the branch points depend on the kernel. They are partially listed in the next lemma, where the equations of the hyperbolas are omitted.

Lemma 4.2. *The functions $Q_1^u(p)$ and $Q_2^u(p)$ are analytic in the complex plane cut along $(-\infty, p_1^u] \cup [p_2^u, \infty)$, where the branch points $p_1^u < 0$ and $p_2^u > 0$ are the real roots of the equation*

$$(\rho^2 - \sigma_1\sigma_2)p^2 + 2(\sigma_2\mu_1 - \rho\mu_2)p + \mu_2^2 = 0. \tag{4.14}$$

The branches Q_1^u and Q_2^u are separated and satisfy

$$\begin{cases} \Re(Q_1^u(ix)) \leqslant 0 \leqslant \Re(Q_2^u(ix)), \quad \forall x \in \mathbb{R}, \\ \Re(Q_1^u(p)) \leqslant \Re(Q_2^u(p)), \quad \forall p \in \mathbb{C}, \\ Q_1^u(0) = \min\left\{0, \dfrac{-2\mu_2}{\sigma_2}\right\}, \quad Q_2^u(0) = \max\left\{0, \dfrac{-2\mu_2}{\sigma_2}\right\}. \end{cases} \tag{4.15}$$

They map the cut $(-\infty, p_1^u]$ (resp. $[p_2^u, \infty)$) onto the right branch $\mathcal{K}_+^u$ (resp. the left branch $\mathcal{K}_-^u$) of the hyperbola $\mathcal{K}^u$.

Similarly, the functions $Q_1^v(p)$ and $Q_2^v(p)$ are analytic in the complex plane cut along $(-\infty, p_1^v] \cup [p_2^v, \infty)$, where the branch points $p_1^v < 0$ and $p_2^v > 0$ are the real roots of the equation

$$(\rho^2 - \sigma_1\sigma_2)p^2 + 2(\sigma_1\mu_2 - \rho\mu_1)p + \mu_1^2 = 0. \tag{4.16}$$

The branches Q_1^v and Q_2^v are separated and satisfy

$$\begin{cases} \Re(Q_1^v(ix)) \leqslant 0 \leqslant \Re(Q_2^v(ix)), \quad \forall x \in \mathbb{R}, \\ \Re(Q_1^v(p)) \leqslant \Re(Q_2^v(p)), \quad \forall p \in \mathbb{C}, \\ Q_1^v(0) = \min\left\{0, \dfrac{-2\mu_1}{\sigma_1}\right\}, \quad Q_2^v(0) = \max\left\{0, \dfrac{-2\mu_1}{\sigma_1}\right\}. \end{cases} \tag{4.17}$$

They map the cut $(-\infty, p_1^v]$ (resp. $[p_2^v, \infty)$) onto the right branch $\mathcal{K}_+^v$ (resp. the left branch $\mathcal{K}_-^v$) of the hyperbola $\mathcal{K}^v$.

It is worth remarking at once that, by using (4.3), (4.4) and Lemma 4.1, one can set *two boundary value problems* for the couple of functions $[\ell_1(p), \ell_2(p)]$ on the respective hyperbolas $\mathcal{H}_+^u$ and $\mathcal{H}_+^v$.

4.3. **Meromorphic continuation to the complex plane.** The method relies on an iterative algorithm, as in [11, Chap. 10], and the following theorem holds. Below and throughout, if $\mathcal{H}_\pm$ denotes a branch of hyperbola as on Figure 4.1, $\mathcal{H}_{\pm,\text{int}}$ will represent the left connected component of $\mathbb{C} \setminus \mathcal{H}_{\pm,\text{int}}$. A similar notation holds for $\mathcal{K}_{\pm,\text{int}}$.

Theorem 4.3. *The functions m, n, ℓ_1 and ℓ_2 can be continued as meromorphic functions to the whole complex plane cut along proper positive real half-lines in their respective planes. The number of poles is finite, and the possible poles of m and n inside the domain $\mathcal{K}_{-,\text{int}}^u \bigcup \mathcal{H}_{-,\text{int}}^v$ coincide.*

4.4. **Reduction to a vectorial Hilbert boundary value problem.** For all $q \in (-\infty, q_1]$, Equations (4.3) and (4.4) yield the linear system

$$\begin{cases} A(P_1^u(q), q)m(q) + \theta n(q) + C(P_1^u(q), q)\ell_1(P_1^u(q)) + E = 0, \\ B(P_1^v(q), q)m(q) - \theta n(q) + D(P_1^v(q), q)\ell_2(P_1^v(q)) - E = 0, \end{cases}$$

which in turn gives

$$\begin{cases} m(q) = \dfrac{C(P_1^u(q), q)\ell_1(P_1^u(q)) + D(P_1^v(q), q)\ell_2(P_1^v(q))}{\Delta(q)}, \\ n(q) = \dfrac{B(P_1^v(q), q)C(P_1^u(q), q)\ell_1(P_1^u(q)) - A(P_1^u(q), q)D(P_1^v(q), q)\ell_2(P_1^v(q))}{\theta\Delta(q)} - \dfrac{E}{\theta}, \end{cases} \tag{4.18}$$

where

$$\Delta(q) \stackrel{\text{def}}{=} -\big(A(P_1^u(q), q) + B(P_1^v(q), q)\big) = -\theta\big(q + P_1^u(q) + P_1^v(q)\big). \tag{4.19}$$

Now, by using the continuity of the left-hand side of the system (4.18) when q traverses the cut $(-\infty, q_1]$, we can set a two-dimensional homogeneous Hilbert-boundary value problem for the vector $[\ell_1, \ell_2]$. More precisely, we first deduce from (4.18) the two following relations, which hold for all $q \in (-\infty, q_1]$:

$$\frac{C(P_1^u(q),q)\ell_1(P_1^u(q)) + D(P_1^v(q),q)\ell_2(P_1^v(q))}{\Delta(q)} = \frac{C(\overline{P_1^u(q)},q)\ell_1(\overline{P_1^u(q)}) + D(\overline{P_1^v(q)},q)\ell_2(\overline{P_1^v(q)})}{\overline{\Delta(q)}}, \tag{4.20}$$

and

$$\frac{C(P_1^u(q),q)B(P_1^v(q),q)\ell_1(P_1^u(q)) - A(P_1^u(q),q)D(P_1^v(q),q)\ell_2(P_1^v(q))}{\Delta(q)} = \frac{B(\overline{P_1^v(q)},q)C(\overline{P_1^u(q)},q)\ell_1(\overline{P_1^u(q)}) - A(\overline{P_1^u(q)},q)D(\overline{P_1^v(q)},q)\ell_2(\overline{P_1^v(q)})}{\overline{\Delta(q)}}. \tag{4.21}$$

Introducing the vector $L(q) \stackrel{\text{def}}{=} [\ell_1(P_1^u(q)), \ell_2(P_1^v(q))]$ and the 2×2-matrix

$$G(q) \stackrel{\text{def}}{=} \frac{1}{\overline{\Delta(q)}} \begin{pmatrix} \dfrac{-\bar\gamma(\alpha+\bar\beta)}{\gamma} & \dfrac{\bar\delta(\bar\alpha-\alpha)}{\gamma} \\ \dfrac{\bar\gamma(\bar\beta-\beta)}{\delta} & \dfrac{-\bar\delta(\beta+\bar\alpha)}{\delta} \end{pmatrix}, \tag{4.22}$$

with

$$\alpha = A(P_1^u(q),q), \quad \beta = B(P_1^v(q),q), \quad \gamma = C(P_1^u(q),q), \quad \delta = D(P_1^v(q),q), \tag{4.23}$$

the system (4.20)–(4.21) immediately yields the following result:

Theorem 4.4. *We have*

$$L^+(q) = G(q)L^-(q), \quad \forall q \in (-\infty, q_1],$$

where $L^+(q)$ (resp. $L^-(q)$) is the limit of $L(q)$ when q reaches the cut from below (resp. from above) in the complex plane.

Remark 4.5. The determinant of the matrix in (4.22) has modulus one, and it is interesting to ask whether this fact could be anticipated. More precisely, with our notation (4.23), the determinant can be rewritten as

$$-\frac{\overline{\gamma\delta}}{\gamma\delta}\frac{\alpha+\beta}{\overline{\alpha+\beta}}.$$

Let us denote by ω_u the conformal mapping of $\mathcal{H}^u_{+,\text{int}}$ onto the unit disk $\mathcal{D}$. Then ω_u is analytic in $\mathcal{H}^u_{+,\text{int}}$, its inverse function ω_u^{-1} is analytic in $\mathcal{D}$, and we have

$$|\omega_u(p)| = 1, \quad \forall p \in \mathcal{H}^u_+.$$

Moreover, by symmetry, one can choose $\omega_u(\overline{p}) = \overline{\omega_u(p)}$, $\forall p \in \mathcal{H}^u_+$, so that

$$\omega_u(\overline{p}) = \frac{1}{\omega_u(p)}.$$

For $|z| = 1$ with $z = A(P_1^u(q), q)$, we have $\overline{z} = 1/z$ and $\overline{p} = \omega_u^{-1}(1/z)$. Similar definitions hold by exchanging the roles of u and v.

Then, setting

$$\Phi^+(z) \stackrel{\text{def}}{=} [\ell_1(\omega_u^{-1}(z)), \ell_2(\omega_v^{-1}(z))], \quad \forall z \in \mathcal{D},$$

and

$$\Phi^-(z) \stackrel{\text{def}}{=} \Phi^+(1/z), \quad \forall |z| > 1,$$

we obtain the boundary condition

$$\Phi^+(z) = H(z)\Phi^-(z), \quad \forall |z| = 1, \tag{4.24}$$

where $H(z)$ is the 2×2 matrix directly derived from $G(q)$, given in (4.22), by using the functions $\omega_u(p)$ and $\omega_v(p)$.

The problem can now be formulated as follows: *Find a sectionally meromorphic vector $\Phi(z)$, constant at infinity, equal to $\Phi^+(z)$ (resp. $\Phi^-(z)$) for $z \in \mathcal{D}$ (resp. for $z \notin \mathcal{D}$), and which satisfies the boundary condition* (4.24).

4.5. **On the solvability of the vectorial boundary value problem** (4.24)**.** It is natural to ask whether the boundary value problem (4.24) may be solved in closed form. As a matter of comparison, scalar (i.e., one-dimensional) boundary value problems may be solved in terms of contour integrals; this is the situation encountered in the convex case [1, 17] as well as in the non-convex symmetric case, as shown in Section 5. However, in the theory, vectorial boundary value problems are in general not solvable in closed form [26, 35, 12]. Accordingly, in the generic non-symmetric case, we are not able to solve completely and explicitly our boundary value problem (4.24). This will be the topic of future research. Let us however do a few remarks:

- To the best of our knowledge, the only asymmetric case which admits a density in closed form is the one mentioned at the end of the introduction, with explicit formula (1.1), see Figure 1.4. This example works for any opening angle $\beta \in (0, 2\pi)$. This explicit example is not obtained as a consequence of the vectorial problem (4.24), but rather from an analogy with the convex case studied in [3]. However, a direct (but tedious) computation could allow us to verify that the vectorial boundary value problem (4.24) is indeed satisfied in this case.
- The solvability of (4.24) should be strongly related to potential nice factorizations of the matrix $H(z)$. For example, in case the matrix $H(z)$ could be written as the product of matrices $\Psi^+(z)^{-1}\Psi^-(z)$, with Ψ sectionally meromorphic on the complex plane cut along the unit circle, then (4.24) could be rewritten as the homogeneous problem $(\Psi\Phi)^+(z) = (\Psi\Phi)^-(z)$, which is solvable. Finding such factorizations appears as a kind of vectorial Tutte's invariant method, in the terminology of [16, 3].

- In the case Φ has no pole inside the unit disk, the solution to (4.24) is shown to be directly connected with the Fredholm integral equation (see, e.g., [26, 35]),
$$\Phi^-(z_0) - \frac{1}{2\pi}\int_{|z|=1} \frac{H^{-1}(z_0)H(z) - \mathcal{I}}{z - z_0}\Phi^-(z)\mathrm{d}z = \Phi^-(\infty),$$
where $\mathcal{I}$ stands for the identity matrix.

5. The symmetric case

When the model is symmetric, we shall put
$$\mu \overset{\text{def}}{=} \mu_1 = \mu_2, \quad \sigma \overset{\text{def}}{=} \sigma_1 = \sigma_2 \quad \text{and} \quad r \overset{\text{def}}{=} r_1 = r_2.$$
The invariant measure is symmetric w.r.t. the diagonal $z_1 = z_2$. Consequently, we have $\pi(z_1, z_2) = \pi(z_2, z_1)$, which yields $n(x + y) = 0$, see (3.2).

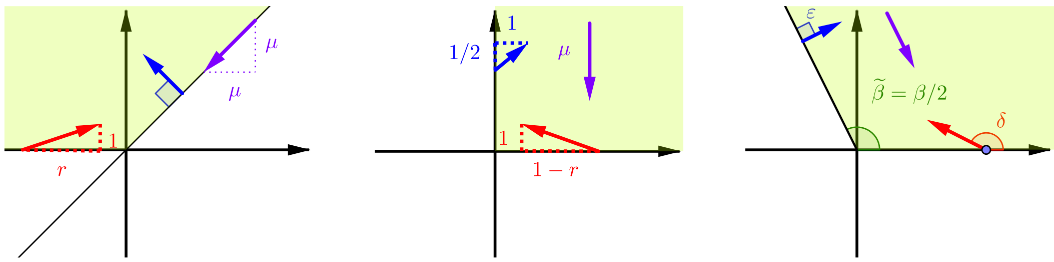

Figure 5.1. In the symmetric case, Brownian motion in a three-quarter plane can be reduced to a more standard reflected Brownian motion in a convex cone. More precisely, its projection in S_1 defines a Brownian motion $\widehat{Z}$ in a wedge of opening $3\pi/8$. The left picture above represents the drift and reflection vectors of $\widetilde{Z}$, to be studied in Section 5.1. After a first change of variable, it becomes a Brownian motion $\widetilde{Z}$ in the quarter plane, as studied in Section 5.2, see the middle picture. On the right, the model is mapped to a $\beta/2$-cone through a linear transform, so as to admit an identity covariance matrix: this last model will be denoted by $T\widetilde{Z}$, see Section 5.4.

5.1. **Reformulation as a reflected Brownian motion in a 3/8 plane.**

Let $\widehat{Z}_t$ be the reflected process of Z_t along the diagonal defined by
$$\widehat{Z}_t \overset{\text{def}}{=} (\widehat{Z}^1_t, \widehat{Z}^2_t) \overset{\text{def}}{=} \frac{1}{2}(Z^1_t + Z^2_t - |Z^2_t - Z^1_t|, Z^1_t + Z^2_t + |Z^2_t - Z^1_t|) = \begin{cases} (Z^1_t, Z^2_t) & \text{if } Z_t \in S_1, \\ (Z^2_t, Z^1_t) & \text{if } Z_t \in S_2. \end{cases}$$
As the following result will establish, the process $\widehat{Z}_t$ is a standard reflected Brownian motion in the convex cone S_1, with reflection vector $(r, 1)$ on the horizontal axis and an orthogonal reflection on the diagonal, see Figure 5.1 (left). We also provide a semimartingale decomposition of this reflected process.

Lemma 5.1. *In the symmetrical case, we have*

$$\begin{cases} \widehat{Z}_t^1 = \widehat{Z}_0^1 + \widehat{W}_t^1 + \mu t + r\widehat{L}_t^1 - \frac{1}{2}\widehat{L}_t^2, \\ \widehat{Z}_t^2 = \widehat{Z}_0^2 + \widehat{W}_t^2 + \mu t + \widehat{L}_t^1 + \frac{1}{2}\widehat{L}_t^2, \end{cases}$$

where $\widehat{W}_t$ is a Brownian motion with the same covariance matrix as W_t, $\widehat{L}_t^2$ is the local time of $\widehat{Z}_t$ on the diagonal, and $\widehat{L}_t^1 = L_t^1 + L_t^2$ is the local time of $\widehat{Z}_t$ on the horizontal axis. We deduce that $\widehat{Z}$ is a reflected Brownian motion in a $3/8$-plane, with reflection vector $(r,1)$ on the horizontal axis and an orthogonal reflection on the diagonal.

Proof. By (2.2), we have

$$Z_t^2 - Z_t^1 = Z_0^2 - Z_0^1 + W_t^2 - W_t^1 + (r-1)(L_t^2 - L_t^1).$$

We apply ItÃt'-Tanaka formula (see Theorem 1.5 in [31, Chap. VI §1]) to the continuous semimartingale $Z_t^2 - Z_t^1$ and to the absolute value $|\cdot|$. We obtain

$$\begin{aligned} |Z_t^2 - Z_t^1| = Z_0^2 - Z_0^1 &+ \int_0^t \operatorname{sgn}(Z_t^2 - Z_t^1)(\mathrm{d}W_t^2 - \mathrm{d}W_t^1) \\ &+ (r-1)\int_0^t \operatorname{sgn}(Z_t^2 - Z_t^1)(\mathrm{d}L_t^2 - \mathrm{d}L_t^1) + \widehat{L}_t^2 \\ = Z_0^2 - Z_0^1 &+ \int_0^t \operatorname{sgn}(Z_t^2 - Z_t^1)(\mathrm{d}W_t^2 - \mathrm{d}W_t^1) + (1-r)(L_t^1 + L_t^2) + \widehat{L}_t^2, \end{aligned}$$

as L_t^1 increases only when $Z_t^2 - Z_t^1 > 0$ and L_t^2 increases only when when $Z_t^2 - Z_t^1 < 0$. Let us recall that, by definition, $\widehat{L}_t^1 = L_t^1 + L_t^2$ and remark that by (2.2), we have

$$Z_t^1 + Z_t^2 = Z_0^1 + Z_0^2 + W_t^1 + W_t^2 + 2\mu t + (r+1)(L_t^1 + L_t^2).$$

Then, we directly obtain that

$$\begin{cases} \widehat{Z}_t^1 = \frac{1}{2}(Z_t^1 + Z_t^2 - |Z_t^2 - Z_t^1|) = \widehat{Z}_0^1 + \widehat{W}_t^1 + \mu t + r\widehat{L}_t^1 - \frac{1}{2}\widehat{L}_t^2, \\ \widehat{Z}_t^2 = \frac{1}{2}(Z_t^1 + Z_t^2 + |Z_t^2 - Z_t^1|) = \widehat{Z}_0^2 + \widehat{W}_t^2 + \mu t + \widehat{L}_t^1 + \frac{1}{2}\widehat{L}_t^2, \end{cases}$$

where we defined

$$\begin{cases} \widehat{W}_t^1 \overset{\text{def}}{=} \int_0^t \frac{1+\operatorname{sgn}(Z_t^2 - Z_t^1)}{2}\mathrm{d}W_t^1 + \int_0^t \frac{1-\operatorname{sgn}(Z_t^2 - Z_t^1)}{2}\mathrm{d}W_t^2, \\ \widehat{W}_t^2 \overset{\text{def}}{=} \int_0^t \frac{1-\operatorname{sgn}(Z_t^2 - Z_t^1)}{2}\mathrm{d}W_t^1 + \int_0^t \frac{1+\operatorname{sgn}(Z_t^2 - Z_t^1)}{2}\mathrm{d}W_t^2. \end{cases}$$

We easily verify that the associated quadratic variations satisfy $\langle \widehat{W}^1 \rangle_t = \langle W^1 \rangle_t = \sigma_1 t$, $\langle \widehat{W}^2 \rangle_t = \langle W^2 \rangle_t = \sigma_2 t$ and $\langle \widehat{W}^1, \widehat{W}^2 \rangle_t = \langle W^1, W^2 \rangle_t = \rho t$ and we conclude by LÃl'vy's characterization theorem, see Theorem 3.6 in [31, Chap. IV §3 p150]. ∎

The reflected process $\widehat{Z}$ is also recurrent and we denote $\widehat{\pi}$ its stationary distribution.

Proposition 5.2. *For all measurable $A \subset S_1$, we have $\pi(A) = \frac{1}{2}\widehat{\pi}(A)$.*

Proof. Let $A \subset S_1$ and $\widehat{A} \in S_2$ be the symmetric set with respect to the first diagonal. In the symmetric case, we have $\pi(A) = \pi(\widehat{A})$. By the ergodic properties of an invariant measure we have $\pi(A) = \lim_{t\to\infty} \mathbb{P}[Z_t \in A]$. Then

$$\begin{aligned}\pi(A) &= \frac{1}{2}(\pi(A) + \pi(\widehat{A}))\\ &= \frac{1}{2}\lim_{t\to\infty}\Big(\mathbb{P}[Z_t \in A] + \mathbb{P}[Z_t \in \widehat{A}]\Big)\\ &= \frac{1}{2}\lim_{t\to\infty}\mathbb{P}[\widehat{Z}_t \in A]\\ &= \frac{1}{2}\widehat{\pi}(A).\end{aligned}$$

■

5.2. **Reformulation as a reflected Brownian motion in a quarter plane.** We now perform a change of variable to obtain a new process $\widetilde{Z}_t$ in the positive quarter plane defined by

$$\widetilde{Z}_t \overset{\text{def}}{=} (-\widehat{Z}^1_t + \widehat{Z}^2_t, \widehat{Z}^2_t),$$

see Figure 5.1. This reformulation at hand, we will be able to use the numerous results in the literature on reflected Brownian motion in a quadrant. Let us emphasize here that our drift is vertical (as shown below), while most of the existing results actually assume that the drift is either zero or oblique (with two non-zero coordinates). Accordingly, some attention is needed when applying directly previous results.

Proposition 5.3. *The process $\widetilde{Z}_t$ satisfies*

$$\begin{cases}\widetilde{Z}^1_t = \widetilde{Z}^1_0 + \widetilde{W}^1_t + (1-r)\widehat{L}^1_t + \widehat{L}^2_t,\\ \widetilde{Z}^2_t = \widetilde{Z}^2_0 + \widetilde{W}^2_t + \mu t + \widehat{L}^1_t + \frac{1}{2}\widehat{L}^2_t,\end{cases}$$

where $\widetilde{W}$ is a Brownian motion with covariance matrix

$$\widetilde{\Sigma} \overset{\text{def}}{=} \begin{pmatrix} \widetilde{\sigma}_1 & \widetilde{\rho}\\ \widetilde{\rho} & \widetilde{\sigma}_2\end{pmatrix} = \begin{pmatrix} 2(\sigma-\rho) & (\sigma-\rho)\\ (\sigma-\rho) & \sigma\end{pmatrix},$$

while $\widehat{L}^1_t$ is the local time of the process on the horizontal axis and $\widehat{L}^2_t$ is the local time on the vertical axis. Thus $\widetilde{Z}_t$ is a reflected Brownian motion in the quadrant $\mathbb{R}^2_+$ with drift $(0,\mu)$, covariance matrix $\widetilde{\Sigma}$ and reflections $(\widetilde{r}_1, 1) \overset{\text{def}}{=} (1-r, 1)$ and $(1, \widetilde{r}_2) \overset{\text{def}}{=} (1, 1/2)$.

Proof. By Lemma 5.1, we have

$$\begin{cases}\widehat{Z}^1_t = \widehat{Z}^1_0 + \widehat{W}^1_t + \mu t + r\widehat{L}^1_t - \frac{1}{2}\widehat{L}^2_t,\\ \widehat{Z}^2_t = \widehat{Z}^2_0 + \widehat{W}^2_t + \mu t + \widehat{L}^1_t + \frac{1}{2}\widehat{L}^2_t,\end{cases}$$

where $\widehat{W}_t$ is a Brownian motion with the same covariance matrix as W_t, $\widehat{L}^2_t$ is the local time of $\widehat{Z}_t$ on the diagonal and $\widehat{L}^1_t = L^1_t + L^2_t$ is the local time of $\widehat{Z}_t$ on the horizontal axis. Then we have

$$\begin{cases}\widetilde{Z}^1_t = -\widehat{Z}^1_0 + \widehat{Z}^2_0 - \widehat{W}^1_t + \widehat{W}^2_t + (1-r)\widehat{L}^1_t + \widehat{L}^2_t,\\ \widetilde{Z}^2_t = \widehat{Z}^2_0 + \widehat{W}^2_t + \mu t + \widehat{L}^1_t + \frac{1}{2}\widehat{L}^2_t.\end{cases}$$

The covariance matrix of the Brownian motion $\widetilde{W}_t \stackrel{\text{def}}{=} (-\widehat{W}_t^1 + \widehat{W}_t^2, \widehat{W}_t^2)$ is

$$\begin{pmatrix} \widetilde{\sigma}_1 & \widetilde{\rho} \\ \widetilde{\rho} & \widetilde{\sigma}_2 \end{pmatrix} = \begin{pmatrix} 2(\sigma-\rho) & (\sigma-\rho) \\ (\sigma-\rho) & \sigma \end{pmatrix}. \qquad \blacksquare$$

Let $\widehat{L}_1(x,y)$ be the Laplace transform of $\frac{1}{2}\widehat{\pi}$ and $\widetilde{L}_1(p,q)$ be the Laplace transform of $\frac{1}{2}\widetilde{\pi}$, where $\widetilde{\pi}$ is the stationary distribution of $\widetilde{Z}$. Let finally be $L_1(x,y)$ the Laplace transform as in (3.1).

Lemma 5.4. *For $(p,q) = (-x, x+y)$, the various Laplace transforms satisfy*

$$L_1(x,y) = \widehat{L}_1(x,y) = \widetilde{L}_1(-x,x+y) = \widetilde{L}_1(p,q).$$

Proof. Proposition 5.2 implies that $L_1(x,y) = \widehat{L}_1(x,y)$. Using that $\widetilde{Z}_t = (-\widehat{Z}_t^1 + \widehat{Z}_t^2, \widehat{Z}_t^2)$, a simple change of variable in the Laplace transform yields $\widehat{L}_1(x,y) = \widetilde{L}_1(-x,x+y)$. $\blacksquare$

5.3. **Functional equations.** We now state a functional equation, which characterizes the Laplace transform $\widetilde{L}_1(p,q)$.

Proposition 5.5. *In the symmetrical case, the following functional equation holds:*

$$U(p,q)\widetilde{L}_1(p,q) + C(p,q)\ell_1(p) + A(p,q)m(q) = 0, \tag{5.1}$$

where

$$\begin{cases} U(p,q) = (\sigma-\rho)p^2 + (\sigma-\rho)qp + \dfrac{\sigma q^2}{2} + \mu q, \\ C(p,q) = (1-r)p + q, \\ A(p,q) = (\sigma-\rho)(p+\frac{1}{2}q). \end{cases} \tag{5.2}$$

As a consequence of Proposition 5.5, the Laplace transform $\widetilde{L}_1(p,q)$ may be computed along the same way as in [17] (contour integral expressions) or [3] (hypergeometric expressions). Interestingly, this functional equation may be obtained by two different techniques:

1. We can use the functional equation (4.3) already obtained in the general (a priori non-symmetric) case and apply it to the symmetric case, using Lemma 5.4.
2. We can also use Proposition 5.3, which says that $\widetilde{Z}$ is a reflected Brownian motion in a quadrant and use the functional equation already known in the bibliography [5, Eq. (2.3)] and [17, Eq. (5)].

We present both proofs below.

Proof 1 (of Proposition 5.5). In the symmetric case, the main functional equation (see Proposition 3.1) takes the simpler form

$$K(x,y)L_1(x,y) + k(x,y)m(x+y) + k_1(x,y)\ell_1(x) = 0, \tag{5.3}$$

where

$$K(x,y) = \frac{1}{2}(\sigma x^2 + 2\rho xy + \sigma y^2) + \mu(x+y),$$

and

$$k(x,y) = \frac{1}{2}(\sigma-\rho)(-x+y) \quad \text{and} \quad k_1(x,y) = rx+y.$$

As in Section 4.2, we introduce the new variables

$$p = -x \quad \text{and} \quad q = x+y.$$

Keeping the same names for the unknown functions, we get from (5.3) and (4.3)

$$U(p,q)L_1(p,q) + C(p,q)\ell_1(p) + A(p,q)m(q) = 0,$$

where, by using (4.5), we obtain the value of U, C and A given in (5.2). ■

Proof 2 (of Proposition 5.5). By Proposition 5.3, the process $\widetilde{Z}$ is a reflected Brownian motion in a quadrant. We denote by $\widetilde{\nu}$ the density of the boundary invariant measure of $\widetilde{Z}$ on the vertical axis, which is defined by

$$\widetilde{\nu}(x)\mathrm{d}x = \mathbb{E}_\Pi \int_0^1 1_{\mathrm{d}x\times\{0\}}(\widetilde{Z}_s)\mathrm{d}\widehat{L}_s^2.$$

Now recall from [3, §2.2] that we have

$$\widetilde{\nu}(x) = (\sigma-\rho)\widetilde{\pi}(0,x) = 2(\sigma-\rho)\pi(x,x).$$

It follows that the Laplace transform of $\widetilde{\nu}$ is equal to $2(\sigma-\rho)m(q)$. It remains to use the well-known functional equation for a reflected Brownian motion in a quadrant, see, e.g., [5, Eq. (2.3)] and [17, Eq. (5)]. Thus, we obtain the functional equation (5.3). ■

5.4. **Reformulation as a reflected Brownian motion in a β-cone.** Let β be the angle in $(\pi, 2\pi)$ such that $\cos\beta = -\rho/\sigma$, that is

$$\beta = 2\pi - \arccos(-\rho/\sigma) \in (\pi, 2\pi).$$

The simple linear mapping

$$T \stackrel{\text{def}}{=} \frac{1}{\sqrt{\sigma}} \begin{pmatrix} \frac{1}{\sin\beta} & \cot\beta \\ 0 & 1 \end{pmatrix}$$

given in the appendix of [17] transforms the reflected Brownian motion Z of covariance matrix Σ in the three-quarter plane into a Brownian motion in a non-convex cone of angle β, with identity covariance matrix and with two equal reflection angles δ such that

$$\tan\delta = \frac{\sin\beta}{r+\cos\beta}. \tag{5.4}$$

Proposition 5.6. *The process $T\widetilde{Z}$ is a reflected Brownian motion in a cone of angle $\beta/2$ and reflection angle $\varepsilon = \pi/2$ and $\delta \in (0,\pi)$ defined in (5.4), see Figure 5.1.*

Proof. The Brownian motion $\widetilde{Z}$ has for covariance matrix

$$\begin{pmatrix} \widetilde{\sigma}_1 & \widetilde{\rho} \\ \widetilde{\rho} & \widetilde{\sigma}_2 \end{pmatrix} = \begin{pmatrix} 2(\sigma-\rho) & (\sigma-\rho) \\ (\sigma-\rho) & \sigma \end{pmatrix},$$

see Proposition (5.3). Let

$$\widetilde{\beta} = \arccos\left(-\frac{\widetilde{\rho}}{\sqrt{\widetilde{\sigma}_1\widetilde{\sigma}_2}}\right) = \arccos\left(-\sqrt{\frac{1}{2}\left(1-\frac{\rho}{\sigma}\right)}\right)$$

the angle associated to the new kernel U. In particular, $\widetilde{\beta} \in (\frac{\pi}{2}, \pi)$, and we have

$$\cos^2 \widetilde{\beta} = \frac{1+\cos\beta}{2},$$

whence

$$\cos\beta = \cos 2\widetilde{\beta} \quad \text{and} \quad \beta = 2\widetilde{\beta},$$

see also [33, Lem. 10] and [27]. Then the new reflection matrix is equal to

$$\begin{pmatrix} 1 & \widetilde{r}_2 \\ \widetilde{r}_1 & 1 \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 1-r \\ 1/2 & 1 \end{pmatrix}.$$

Performing the same change of variables as in the appendix of [17], this equation amounts to studying a Brownian motion in a wedge of angle $\widetilde{\beta}$, identity covariance matrix and reflection angles

$$\tan\varepsilon = \frac{\sin\widetilde{\beta}}{\widetilde{r}_1\sqrt{\widetilde{\sigma}_1/\widetilde{\sigma}_2} + \cos\widetilde{\beta}} \quad \text{and} \quad \tan\delta = \frac{\sin\widetilde{\beta}}{\widetilde{r}_2\sqrt{\widetilde{\sigma}_2/\widetilde{\sigma}_1} + \cos\widetilde{\beta}}.$$

Then we get

$$\tan\varepsilon = \infty, \text{ i.e., } \varepsilon = \pi/2 \quad \text{and} \quad \tan\delta = \frac{2\cos\widetilde{\beta}\sin\widetilde{\beta}}{r-1+2\cos^2\widetilde{\beta}} = \frac{\sin\beta}{r+\cos\beta}. \tag{5.5}$$ ■

5.5. **Algebraic nature of the Laplace transform.** For reflected Brownian motion in a quadrant, the work [3] proposes an exhaustive classification of the parameters (drift, opening of the cone and reflection angles), allowing to decide which of the following classes of functions the associated Laplace transform $\widetilde{L}_1(p,q)$ belongs to:

(C1) Rational
(C2) Algebraic
(C3) D-finite (D for Differentially) (by this, we mean that the Laplace transform satisfies two linear differential equations with coefficients in $\mathbb{R}(p,q)$, one in p and one in q)
(C4) D-algebraic (that is, when it satisfies a polynomial differential equation in p, and another in q)
(C5) D-transcendental (when it is non-D-algebraic)

Notice that the classes (C1) to (C4) define a hierarchy, in the sense that

$$\text{(C1)} \subset \text{(C2)} \subset \text{(C3)} \subset \text{(C4)}.$$

A more probabilistic description of the models having a Laplace transform in the class (C1) above is as follows:

- The skew symmetric condition: $\varepsilon + \delta = \pi$, which is a necessary and sufficient condition for the stationary distribution to be exponential, see [22].
- The Dieker and Moriarty [7] criterion: $\varepsilon + \delta - \pi \in -\beta\mathbb{N}$, which is a necessary and sufficient condition for the stationary distribution to be a sum of exponential terms.

Accordingly, we may transfer the classification of [3] to our symmetric Brownian motion in a three-quarter plane, via its projection in the domain S_1 and its quadrant description $\widetilde{Z}$. Then the following proposion holds.

Proposition 5.7. *The Laplace transform of the reflected Brownian motion in the quarter plane $\widetilde{Z}$ is never rational (class (C1)). However, there exist values of parameters such that $\widetilde{L}$ is D-algebraic, D-finite or algebraic.*

Before proving Proposition 5.7, let us do some remarks:

- As a consequence, there is no skew symmetry in the three-quarter plane (nor Dieker and Moriarty condition). From this point of view, Brownian motion in non-convex cones is deeply different from Brownian motion in convex cones.
- The above feature (absence of skew symmetry) admits a clear interpretation in terms of the growth of exponential functions in $\mathbb{R}^2$. Indeed, for $(a,b) \neq (0,0)$, an exponential function
$$(p,q) \mapsto \exp(-ap-bq) \tag{5.6}$$
tends to infinity in half of the directions of $\mathbb{R}^2$, so such an exponential function (and any finite linear combination of exponential functions as well) will never be integrable on a non-convex domain. As a direct consequence, it cannot represent any stationary distribution.
- The example presented in Figure 1.4 (see (1.1)) has an algebraic Laplace transform, as computed in [3]. It appears as the simplest example which one may construct in a non-convex wedge.

Proof of Proposition 5.7. The skew symmetric condition is
$$2\widetilde{\rho} = \widetilde{r}_1\widetilde{\sigma}_1 + \widetilde{r}_2\widetilde{\sigma}_2,$$
or
$$\sigma - \rho = (1/2)(\sigma - \rho) + (1-r)\sigma/2,$$
which yields $r = \rho/\sigma < 1$. Hence, as the recurrence conditions imply $r > 1$, we can conclude that the skew symmetric case is not possible. More generally the Dieker and Moriarty condition
$$\varepsilon + \delta - \pi \in -\mathbb{N}\widetilde{\beta}$$
cannot hold, because $\varepsilon + \delta - \pi = \delta - \pi/2 > 0$. However, there exist some parameters such that
$$\pi/2 + \delta \in \widetilde{\beta}\mathbb{Z} + \pi\mathbb{Z},$$
which is exactly condition [3] to admit a D-algebraic Laplace transform. ■

5.6. **Line of steepest descent of π.** In the symmetric case, we remarked that the Laplace transform of the normal derivative of π along the diagonal is zero and then $n(x,y) = 0$, see (3.2). Thus we may formulate the following question, in the non-symmetric case: does there also exist a line (not necessarily the diagonal) along which the normal derivative of π is zero?

Let us consider the steepest descent line of π starting from $(0,0)$. In other words, we consider that π is a potential and we are looking to the field line of grad π passing through $(0,0)$. This defines the curve

$$\mathcal{C} = \{(z_1(t), z_2(t)) : t \in \mathbb{R}_+\},$$

where $(z_1(0), z_2(0)) = (0,0)$ and

$$\begin{cases} z_1'(t) = \dfrac{\partial \pi}{\partial z_1}(z_1(t), z_2(t)), \\ z_2'(t) = \dfrac{\partial \pi}{\partial z_2}(z_1(t), z_2(t)). \end{cases}$$

If we divide the three-quarter plane along this line, we obtain a functional equation with only two unknown functions. We focus on a few examples where the curve $\mathcal{C}$ is a simple half-line:

- In the symmetric case studied in Section 5, the curve $\mathcal{C}$ is simply the first diagonal.
- In the special case of Figure 1.4, the curve $\mathcal{C}$ is the half-line starting from the origin and following the direction of the drift.
- In the quadrant, when the skew symmetric condition is satisfied, the stationary distribution has an exponential density of the form (5.6) (up to a normalization constant), and the curve $\mathcal{C}$ is also a half-line of direction $-(a,b)$.

5.7. **The roots of the kernel** $U(p,q)$**.** The formulas of Lemmas 4.1 and 4.2 are simplified in a pleasant way.

Lemma 5.8. *The function $U(p,q)$ in (5.1), viewed as a polynomial in the variable q, has two roots $Q_1(p)$ and $Q_2(p)$, which are the branches of a two-sheeted covering over the p-plane. They are analytic in the whole complex plane cut along $(-\infty, p_1] \cup [p_2, \infty)$, with*

$$p_1 = \frac{\mu\big(\sigma - \rho + \sqrt{2\sigma(\sigma-\rho)}\big)}{\sigma^2 - \rho^2} < 0 < p_2 = \frac{\mu\big(\sigma - \rho - \sqrt{2\sigma(\sigma-\rho)}\big)}{\sigma^2 - \rho^2}. \tag{5.7}$$

The branches $Q_1(p)$ and $Q_2(p)$ are separated (except on the cut) and they satisfy

$$\begin{cases} \Re(Q_1(ix)) \leqslant 0 \leqslant \Re(Q_2(ix)), \quad \forall x \in \mathbb{R}, \\ \Re(Q_1(p)) \leqslant \Re(Q_2(p)), \quad \forall p \in \mathbb{C}. \end{cases} \tag{5.8}$$

Proof. The last property of (5.8) is a direct application of the maximum modulus principle to the function $\exp Q(p)$. The proof of the lemma is complete. ■

Mutatis mutandis, the following lemma holds, with the convenient notation.

Lemma 5.9. *The function $R(p,q)$, viewed as a polynomial in the variable p, has two roots $P_1(q)$ and $P_2(q)$, which are the branches of a two-sheeted covering over the q-plane. They are analytic in the whole complex plane cut along $(-\infty, q_1] \cup [q_2, \infty)$, with*

$$q_1 = 0 < q_2 = -\frac{4\mu}{\sigma+\rho}. \tag{5.9}$$

They are separated and satisfy

$$\begin{cases} \Re(P_1(ix)) \leqslant 0 \leqslant \Re(P_2(ix)), & \forall x \in \mathbb{R}, \\ \Re(P_1(p)) \leqslant \Re(P_2(p)), & \forall p \in \mathbb{C}. \end{cases} \tag{5.10}$$

With the above definitions, when $\mu < 0$,

$$P_1(0) = P_2(0) = 0 \quad \text{and} \quad Q_1(0) = 0.$$

Our goal is to set a boundary value problem (BVP) for either of the functions $m(q)$ or $\ell_1(p)$ on an adequate hyperbola.

5.8. **The hyperbolas.** The following lemma is an immediate application of the results of Lemma 4.1.

Lemma 5.10. *The functions Q_1 and Q_2 map the cut $(-\infty, p_1]$ (resp. $[p_2, \infty)$) onto the right branch $\mathcal{H}_q^+$ (resp. the left branch $\mathcal{H}_q^-$) of the hyperbola $\mathcal{H}_q$*

$$(\sigma+\rho)x^2 - (\sigma-\rho)y^2 + 4\mu x + \frac{2\mu^2}{\sigma} = 0, \tag{5.11}$$

rewritten in the canonical form (since $\sigma > |\rho|$) as

$$\left(x + \frac{2\mu}{\sigma+\rho}\right)^2 - \left(\frac{\sigma-\rho}{\sigma+\rho}\right) y^2 = \frac{2\mu^2(\sigma-\rho)}{\sigma(\sigma+\rho)^2}. \tag{5.12}$$

Similarly, P_1 and P_2 map the cut $(-\infty, q_1]$ (resp. $[q_2, \infty)$) onto the right branch $\mathcal{H}_p^+$ (resp.the left branch $\mathcal{H}_p^-$) of the hyperbola $\mathcal{H}_p$

$$\left(x - \frac{\mu}{\sigma+\rho}\right)^2 - \left(\frac{\sigma-\rho}{\sigma+\rho}\right) y^2 = \left(\frac{\mu}{\sigma+\rho}\right)^2, \tag{5.13}$$

which goes through the point $(0,0)$.

Proof. On the cuts $[p_1, \infty)$ and $(-\infty, p_2]$, the quantities $Q_1(p)$ and $Q_2(p)$ take complex conjugate values of the form $x \pm iy$, where

$$Q_1(p) + Q_2(p) = \frac{-2[\mu + (\sigma-\rho)p]}{\sigma} = 2x,$$
$$Q_1(p)Q_2(p) = \frac{2(\sigma-\rho)p^2}{\sigma} = x^2 + y^2.$$

Equations (5.11) and (5.12) follow immediately, and (5.13) is obtained in an entirely similar way. ■

5.9. **Analytic continuation and BVP.** For any arbitrary simple closed curve $\mathcal{U}$, $G_{\mathcal{U}}$ (resp. $G^c_{\mathcal{U}}$) will denote the interior (resp. exterior) domain bounded by $\mathcal{U}$, i.e., the domain remaining on the left-hand side when $\mathcal{U}$ is traversed in the positive (counterclockwise) direction. This definition remains valid for the case when $\mathcal{U}$ is unbounded but closable at infinity. For instance, $G_{\mathcal{H}_q^+}$ (resp. $G^c_{\mathcal{H}_q^+}$) is the region situated to the right (resp. to the left) of the branch $\mathcal{H}_q^+$ of the hyperbola $\mathcal{H}_q$.

Corollary 5.11.

1. *$G_{\mathcal{H}_p^-} \setminus [-\infty, p_1] \underset{P_1(q)}{\overset{Q_2(p)}{\rightleftarrows}} G_{\mathcal{H}_q^+} \setminus [q_2, +\infty]$ and the mappings are conformal.*
2. *The values of Q_1 belong to $G^c_{\mathcal{H}_q^+}$.*
3. *The values of Q_2 belong to $G^c_{\mathcal{H}_q^-}$.*

Moreover, the following automorphy relationships hold:

$$P_1 \circ Q_1(p) = \begin{cases} p, & \text{if } p \in G^c_{\mathcal{H}_p^+}, \\ \neq p, & \text{if } p \in G_{\mathcal{H}_p^+}. \end{cases} \quad \text{Then} \quad P_1 \circ Q_1(G^c_{\mathcal{H}_p^+}) = G^c_{\mathcal{H}_p^+}.$$

$$P_2 \circ Q_1(p) = \begin{cases} p, & \text{if } p \in G_{\mathcal{H}_p^+}, \\ \neq p, & \text{if } p \in G^c_{\mathcal{H}_p^+}. \end{cases} \quad \text{Then} \quad P_2 \circ Q_1(G_{\mathcal{H}_p^+}) = G_{\mathcal{H}_p^+}.$$

$$P_1 \circ Q_2(p) = \begin{cases} p, & \text{if } p \in G_{\mathcal{H}_p^-}, \\ \neq p, & \text{if } p \in G^c_{\mathcal{H}_p^-}. \end{cases} \quad \text{Then} \quad P_1 \circ Q_2(G_{\mathcal{H}_p^-}) = G_{\mathcal{H}_p^-}.$$

$$P_2 \circ Q_2(p) = \begin{cases} p, & \text{if } p \in G^c_{\mathcal{H}_p^-}, \\ \neq p & \text{if } p \in G_{\mathcal{H}_p^-}. \end{cases} \quad \text{Then} \quad P_2 \circ Q_2(G^c_{\mathcal{H}_p^-}) = G^c_{\mathcal{H}_p^-}.$$

Proof. The arguments are analogous to those presented in [11, Chap. 5 and Chap. 6]. Assertion 1 is immediate. As for assertions 2 and 3, they follow mainly from the maximum modulus principle applied to the functions $Q_1(p)$ and $Q_2(p)$ respectively. The automorphy relationships can be checked up to some tedious calculus (omitted). They also can be verified by using the following GeoGebra numerical animation https://www.geogebra.org/m/phvjk35w ■

Letting q tend successively to the upper and lower edge of the slit $(-\infty, q_1]$, and using the fact that $m(q)$ is analytic in the left half-plane $\{\Re(q) \leq 0\}$, we eliminate $m(q)$ from (5.1) to get

$$\ell_1(P_1(q))F(P_1(q), q) - \ell_1(P_2(q))F(P_2(q), q) = 0, \quad \text{for } q \in (-\infty, q_1], \tag{5.14}$$

where

$$F(p, q) = \frac{C(p, q)}{A(p, q)}.$$

Then the determination of $\ell_1(p)$, meromorphic in the domain $G^c_{\mathcal{H}_p^+}$, is equivalent to solving a BVP of Riemann-Hilbert-Carleman type, on the contour $\mathcal{H}_p^+$ in the complex plane, as originally proposed in [10]. More precisely, by using the first two properties of Corollary 5.11, and remembering that on the cut $(-\infty, q_1], P_1(q) = \overline{P_2(q)}$, this BVP takes the following form:

$$\ell_1(p)K(p) - \ell_1(\overline{p})K(\overline{p}) = 0, \quad p \in \mathcal{H}_p^+, \tag{5.15}$$

where $K(p) = F(p, Q_1(p))$, and ℓ_1 is sought to be meromorphic inside $G^c_{\mathcal{H}_p^+}$, its poles being the possible zeros of $C(p, Q_1(p))$ in the region $G^c_{\mathcal{H}_p^+} \cap \{\Re(p) > 0\}$.

Interestingly, Corollary 5.11 allows to carry out the analytic continuation of the functions $\ell_1(p)$ and $m(q)$, satisfying equation (5.1).

Theorem 5.12. *The functional equation*

$$\ell_1(p)F(p, Q_1(p)) - \ell_1(P_2 \circ Q_1(p))F(P_2 \circ Q_1(p), Q_1(p)) = 0 \tag{5.16}$$

is valid for all $p \in \mathbb{C}$ and provides the analytic continuation of ℓ_1 as a meromorphic function (the number of poles being finite) to the whole complex plane cut along $[p_2, \infty)$.

Proof. It is a direct consequence of the automorphy properties given in Corollary 5.11. Indeed, it suffices in equation (5.14) to let q quit the cut $(-\infty, q_1]$, while remaining in $\mathcal{H}_q^-$. Then to this q corresponds a point $p \in G^c_{\mathcal{H}_q^+}$ satisfying $P_1 \circ Q_1(p) = p$, which leads to equation (5.16). ∎

6. Concluding remarks

This preliminary work appeals further developments, especially with regard to the solution of the vector boundary value problem obtained in Section 4.4. A delicate technical issue is the factorization of the matrix (4.22). In this respect, an interesting intermediate semi-symmetrical situation takes place when $\mu_1 = \mu_2$, and $\sigma_1 = \sigma_2$, but $r_1 \neq r_2$, which should lead to some reasonably explicit results!

References

[1] F. Baccelli and G. Fayolle. Analysis of models reducible to a class of diffusion processes in the positive quarter plane. *SIAM J. Appl. Math.*, 47(6):1367–1385, 1987.

[2] M. Bousquet-Mélou. Square lattice walks avoiding a quadrant. *J. Combin. Theory Ser. A*, 144:37–79, 2016.

[3] M. Bousquet-Mélou, A. Elvey Price, S. Franceschi, C. Hardouin, and K. Raschel. The stationary distribution of the reflected Brownian motion in a wedge: differential properties. *arXiv:2101.01562*, 2021.

[4] J. G. Dai and J. M. Harrison. Reflected Brownian motion in an orthant: numerical methods for steady-state analysis. *Ann. Appl. Probab.*, 2(1):65–86, 1992.

[5] J. G. Dai and M. Miyazawa. Reflecting Brownian motion in two dimensions: exact asymptotics for the stationary distribution. *Stoch. Syst.*, 1(1):146–208, 2011.

[6] J. G. Dai and M. Miyazawa. Stationary distribution of a two-dimensional SRBM: geometric views and boundary measures. *Queueing Syst.*, 74(2-3), 2013.

[7] A. B. Dieker and J. Moriarty. Reflected Brownian motion in a wedge: sum-of-exponential stationary densities. *Electron. Commun. Probab.*, 14:1–16, 2009.

[8] A. Elvey Price. Counting lattice walks by winding angle. *Sém. Lothar. Combin.*, 84B:Art. 43, 12, 2020.

[9] P. Ernst and S. Franceschi. Escape and absorption probabilities for obliquely reflected brownian motion in a quadrant. *arXiv:2101.01246*, 2021.

[10] G. Fayolle and R. Iasnogorodski. Two coupled processors: the reduction to a Riemann-Hilbert problem. *Z. Wahrsch. Verw. Gebiete*, 47(3):325–351, 1979.

[11] G. Fayolle, R. Iasnogorodski, and V. Malyshev. *Random walks in the quarter plane*, volume 40 of *Probability Theory and Stochastic Modelling*. Springer, Cham, second edition, 2017. Algebraic methods, boundary value problems, applications to queueing systems and analytic combinatorics.

[12] G. Fayolle and K. Raschel. About a possible analytic approach for walks in the quarter plane with arbitrary big jumps. *C. R. Math. Acad. Sci. Paris*, 353(2):89–94, 2015.

[13] M. E. Foddy. *Analysis of Brownian motion with drift, confined to a quadrant by oblique reflection (diffusions, Riemann-Hilbert problem)*. ProQuest LLC, Ann Arbor, MI, 1984. Thesis (Ph.D.)–Stanford University.

[14] G. J. Foschini. Equilibria for diffusion models of pairs of communicating computers—symmetric case. *IEEE Trans. Inform. Theory*, 28(2):273–284, 1982.

[15] S. Franceschi. Green's functions with oblique Neumann boundary conditions in the quadrant. *J. Theoret. Probab.*, to appear, 2021.

[16] S. Franceschi and K. Raschel. Tutte's invariant approach for Brownian motion reflected in the quadrant. *ESAIM Probab. Stat.*, 21:220–234, 2017.

[17] S. Franceschi and K. Raschel. Integral expression for the stationary distribution of reflected Brownian motion in a wedge. *Bernoulli*, 25(4B):3673–3713, 2019.

[18] J. M. Harrison. The diffusion approximation for tandem queues in heavy traffic. *Adv. in Appl. Probab.*, 10(4):886–905, 1978.

[19] J. M. Harrison and M. I. Reiman. On the distribution of multidimensional reflected Brownian motion. *SIAM J. Appl. Math.*, 41(2):345–361, 1981.

[20] J. M. Harrison and M. I. Reiman. Reflected Brownian motion on an orthant. *Ann. Probab.*, 9(2):302–308, 1981.

[21] J. M. Harrison and R. J. Williams. Brownian models of open queueing networks with homogeneous customer populations. *Stochastics*, 22(2):77–115, 1987.

[22] J. M. Harrison and R. J. Williams. Multidimensional reflected Brownian motions having exponential stationary distributions. *Ann. Probab.*, 15(1):115–137, 1987.

[23] D. G. Hobson and L. C. G. Rogers. Recurrence and transience of reflecting Brownian motion in the quadrant. *Math. Proc. Cambridge Philos. Soc.*, 113(2):387–399, 1993.

[24] I. A. Kurkova and Y. M. Suhov. Malyshev's theory and JS-queues. Asymptotics of stationary probabilities. *Ann. Appl. Probab.*, 13(4):1313–1354, 2003.

[25] V. A. Malyšev. Positive random walks and Galois theory. *Uspehi Mat. Nauk*, 26(1(157)):227–228, 1971.

[26] N. I. Muskhelishvili. *Singular integral equations*. Dover Publications, Inc., New York, 1992. Boundary problems of function theory and their application to mathematical physics, Translated from the second (1946) Russian edition and with a preface by J. R. M. Radok, Corrected reprint of the 1953 English translation.

[27] S. Mustapha. Non-D-finite walks in a three-quadrant cone. *Ann. Comb.*, 23(1):143–158, 2019.

[28] K. Ramanan and M. I. Reiman. Fluid and heavy traffic diffusion limits for a generalized processor sharing model. *Ann. Appl. Probab.*, 13(1):100–139, 2003.

[29] K. Raschel and A. Trotignon. On walks avoiding a quadrant. *Electron. J. Combin.*, 26(3):Paper No. 3.31, 34, 2019.

[30] M. I. Reiman. Open queueing networks in heavy traffic. *Math. Oper. Res.*, 9(3):441–458, 1984.

[31] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.

[32] S. E. Shreve. Reflected Brownian motion in the "bang-bang" control of Brownian drift. *SIAM J. Control Optim.*, 19(4):469–478, 1981.
[33] A. Trotignon. Discrete harmonic functions in the three-quarter plane. *Potential Anal.*, to appear, 2021.
[34] S. R. S. Varadhan and R. J. Williams. Brownian motion in a wedge with oblique reflection. *Comm. Pure Appl. Math.*, 38(4):405–443, 1985.
[35] N. P. Vekua. *Systems of singular integral equations.* P. Noordhoff, Ltd., Groningen, 1967. Translated from the Russian by A. G. Gibbs and G. M. Simmons. Edited by J. H. Ferziger.
[36] R. J. Williams. Reflected Brownian motion in a wedge: semimartingale property. *Z. Wahrsch. Verw. Gebiete*, 69(2):161–176, 1985.

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